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Zero-free approximants to derivatives of prestarlike functions

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Abstract

For a prestarlike function f of nonnegative order α, $0 \leq \alpha < 1$, and a close-to-convex function zg of order α, the convolution $g * f'$ is shown to be zero-free in the open unit disk. The result can be applied to a wide spectrum of interesting approximants, including those involving the Cesàro means and Jacobi polynomials. If zg is also prestarlike, then the range of $g * f'$ is shown to be contained in a sector with opening angle strictly less than 2π.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk $\mathbb{D} = \{z : |z| < 1\}$ of the complex plane, and let $\mathcal{S}$ be its subclass consisting of univalent functions. For $\mu < 1$, let $\mathcal{S}^*(\mu)$ and $\mathcal{C}(\mu)$ be the subclasses of $\mathcal{A}$ consisting respectively of starlike and convex functions of order μ defined analytically by

$$f \in \mathcal{S}^*(\mu) \Leftrightarrow \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > \mu, \quad \text{and} \quad f \in \mathcal{C}(\mu) \Leftrightarrow \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \mu.$$

For brevity, denote $\mathcal{C} := \mathcal{C}(0)$ and $\mathcal{S}^* = \mathcal{S}^*(0)$. The closely-related class $\mathcal{K}(\mu)$ of close-to-convex functions of order μ consists of functions $f \in \mathcal{A}$ satisfying

$$\mathrm{Re}\, \frac{zf'(z)}{g(z)} > 0$$

for some $g \in \mathcal{S}^*(\mu)$. Evidently, for $0 \leq \mu < 1$, $\mathcal{C}(\mu) \subset \mathcal{S}^*(\mu) \subset \mathcal{K}(\mu) \subset \mathcal{K} := \mathcal{K}(0) \subseteq \mathcal{S}$.

For $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ in $\mathbb{D}$, the convolution (or Hadamard product) $f * g$ is given by the series $(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k$. The Cesàro means of a given function is of special interest in this paper. It is the convolution between the function with the Cesàro polynomial. Specifically, let σ_n^{β} be the Cesàro polynomial of nonnegative order β defined by

$$\sigma_n^{\beta}(z) = \frac{n!}{(1+\beta)_n} \sum_{k=0}^{n} \frac{(1+\beta)_{n-k}}{(n-k)!} z^k \quad (n \in \mathbb{N}),$$

where $\mathbb{N}$ is the set of positive integers. Here $(a)_k$ denotes the Pochhammer symbol given by $(a)_0 = 1$ and $(a)_k = a(a + 1)_{k-1}$, $k \in \mathbb{N}$. The Cesàro means $\sigma_n^\beta(z, f)$ of order β for a function $f(z) = \sum_{k=0}^\infty a_k z^k$ is

$$\sigma_n^\beta(z, f) := \sigma_n^\beta(z) * f(z) = \frac{n!}{(1 + \beta)_n} \sum_{k=0}^n \frac{(1 + \beta)_{n-k}}{(n - k)!} a_k z^k \quad (n \in \mathbb{N}).$$

The works of [1, 2] elucidated the geometric properties of the Cesàro polynomial.

A function f is said to be zero-free in $\mathbb{D}$ if $f(z) \neq 0$ for all $z \in \mathbb{D}$. The outer functions, which play an important role in the theory of $\mathcal{H}^p$ spaces, are functions of the form

$$F(z) = e^{i\gamma} \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \psi(t)\, dt \right) \quad (z \in \mathbb{D}),$$

where $\gamma \in \mathbb{R}$, $\psi(t) \geq 0$, $\log \psi(t) \in L^1$ and $\psi(t) \in L^p$. It is known [3, 4] that the derivatives of bounded convex functions are outer functions.

Taylor series or its partial sums are of course natural approximants to a given function. However, Barnard *et al.* [5] showed that the Taylor approximants of outer functions can vanish in $\mathbb{D}$, while the Cesàro means of order one for the derivative of convex functions are zero-free. It is therefore [5, 6] natural to investigate the problem of finding a suitable polynomial approximant for a given outer function f that retains the zero-free property of f.

Swaminathan [6] showed the zero-free property of the Cesàro means σ_n^β and polynomial approximants associated with Jacobi polynomials for the derivative of a prestarlike function of a certain order. Prestarlike functions [7] $\mathcal{R}_\mu$ of order μ, $\mu < 1$, consists of functions $f \in \mathcal{A}$ satisfying $f * k_\mu \in \mathcal{S}^*(\mu)$, $k_\mu(z) := z/(1 - z)^{2-2\mu}$, while $\mathcal{R}_1$ consists of functions $f \in \mathcal{A}$ satisfying $\text{Re}(f(z)/z) > 1/2$. Evidently, $\mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$ and $\mathcal{R}_0 = \mathcal{C}$. The works by [8–10] contained interesting exposition on prestarlike functions.

For prestarlike (and convex) functions f, the present work finds approximants derived from the convolution between f' and g, where zg are close-to-convex of nonnegative order. This general result can be widely applied to include a range of interesting polynomial approximants, and thus connects with the earlier works by [5, 6, 11]. Section 3 gives examples of such applications. If zg is also prestarlike, then the range of $g * f'$ is shown to be contained in a sector with opening angle strictly less than 2π.

The following two results will be required.

Lemma 1.1 [7]

   (i) *If $f, g \in \mathcal{R}_\mu$, $\mu \leq 1$, then $f * g \in \mathcal{R}_\mu$.*
  (ii) *If $\mu \leq \beta \leq 1$, then $\mathcal{R}_\mu \subset \mathcal{R}_\beta$.*
 (iii) *If $f \in \mathcal{S}^*(\mu)$ (or $\mathcal{K}(\mu)$) and $g \in \mathcal{R}_\mu$, $\mu < 1$, then $f * g \in \mathcal{S}^*(\mu)$ (or $\mathcal{K}(\mu)$).*
 (iv) *$\mathcal{R}_\mu \subset \mathcal{S}$ if and only if $\mu \leq 1/2$.*

For $0 \leq \alpha < 1$, let $\mathcal{P}(\alpha)$ denote the class of all analytic functions p defined in $\mathbb{D}$ satisfying $p(0) = 1$ and $\text{Re}\, p(z) > \alpha$. Also simply denote by $\mathcal{P} := \mathcal{P}(0)$. The result in [9, Theorem 2.4, p.54] can be expressed in the following form.

Lemma 1.2 [6, Lemma 3, p.120] *Let $\alpha < 1$, and $0 \leq \beta < 1$. If $f \in \mathcal{R}_\alpha$, $g \in \mathcal{S}^*(\alpha)$ and $p \in \mathcal{P}(\beta)$, then there exists $p_1 \in \mathcal{P}(\beta)$ such that $f * gp = (f * g)p_1$.*

2 Main results

Theorem 2.1 *Let $0 \leq \alpha < 1$. If $f \in \mathcal{R}_\alpha$ and $zg \in \mathcal{K}(\alpha)$, then $g * f'$ is zero-free in $\mathbb{D}$.*

Proof It is sufficient to show that $g * f'$ is a product of two zero-free functions in $\mathbb{D}$. Rewrite $z(g * f')(z)$ as

$$z\big(g * f'\big)(z) = z(zg)'(z) * f(z). \tag{1}$$

Since $zg \in \mathcal{K}(\alpha)$, there exists a function $h \in \mathcal{S}^*(\alpha)$ and $p \in \mathcal{P}$ such that $z(zg)'(z) = h(z)p(z)$. Therefore, the expression on the right side of (1) can be written as

$$z(zg)'(z) * f(z) = \big((hp) * f\big)(z).$$

Since $f \in \mathcal{R}_\alpha$, and $h \in \mathcal{S}^*(\alpha)$, Lemma 1.2 yields a $p_1 \in \mathcal{P}$ such that

$$\big((hp) * f\big)(z) = (h * f)(z)p_1(z).$$

Therefore, (1) implies that

$$\big(g * f'\big)(z) = \frac{(h * f)(z)}{z} p_1(z). \tag{2}$$

It also follows from Lemma 1.1(iii) that $h * f \in \mathcal{S}^*(\alpha)$. Since $\mathcal{S}^*(\alpha) \subset \mathcal{S}$ for $0 \leq \alpha < 1$, $(h * f)(z) = 0$ if and only if $z = 0$. Therefore, $(h(z) * f(z))/z$ is zero-free in $\mathbb{D}$. Further, as $\operatorname{Re} p_1(z) > 0$, (2) implies that $g * f'$ is a product of two zero-free functions, and, hence, it is also zero-free in $\mathbb{D}$. □

Lewis [1] proved that $z\sigma_n^\beta \in \mathcal{K}$ for $\beta \geq 1$. Since $\mathcal{R}_0 = \mathcal{C}$, Theorem 2.1 readily yields the following result on the Cesàro means of the derivative of convex functions.

Corollary 2.1 [6, Theorem 2, p.120] *If $f \in \mathcal{C}$, then the function $\sigma_n^\beta(z, f') = (\sigma_n^\beta * f')(z)$ is zero-free in $\mathbb{D}$ for $\beta \geq 1$.*

3 Examples of approximants

For applications of Theorem 2.1, this section looks at several interesting examples of approximants. For $\beta \geq 0$ and $\alpha \in [0, 1)$, define the polynomial

$$\mathcal{G}_{\alpha,\beta}(z) := 1 + \frac{n!}{(1+\beta)_n} \sum_{k=1}^{n} \frac{(1+\beta)_{n-k}(2-2\alpha)_k}{(n-k)!(k+1)!} z^k = \sigma_n^\beta(z, h), \tag{3}$$

where

$$h(z) := \begin{cases} -\frac{\log(1-z)}{z}, & \alpha = 1/2, \\ \frac{(1-z)^{2\alpha-1}-1}{z(1-2\alpha)}, & \alpha \neq 1/2. \end{cases} \tag{4}$$

The function zh is known to be extremal (see [12]) for many problems in the class $\mathcal{C}(\alpha)$. The following result on Cesàro means for convex function of nonnegative order will be required.

Lemma 3.1 [13, Theorem 4.2] *Let $n \in \mathbb{N}$. If $f \in \mathcal{C}(\lambda)$, $1/2 \le \lambda < 1$, and $\beta \ge 0$, then $z\sigma_{n-1}^{\beta} * f \in \mathcal{K}(\lambda)$.*

Corollary 3.1 *Let $1/2 \le \alpha < 1$, $\beta \ge 0$ and $\mathcal{G}_{\alpha,\beta}$ be given by (3). If $f \in \mathcal{R}_{\alpha}$, then $\mathcal{G}_{\alpha,\beta} * f'$ is zero-free in $\mathbb{D}$.*

Proof We show that $z\mathcal{G}_{\alpha,\beta} \in \mathcal{K}(\alpha)$. It follows from (3) that

$$
\begin{aligned}
z\mathcal{G}_{\alpha,\beta}(z) &= z + \frac{n!}{(1+\beta)_n} \sum_{k=1}^{n} \frac{(1+\beta)_{n-k}(2-2\alpha)_k}{(n-k)!(k+1)!} z^{k+1} \\
&= \left(\sum_{k=1}^{n+1} \frac{(1+\beta)_{n-k+1}}{(n-k+1)!} \frac{n!}{(1+\beta)_n} z^k \right) * \left(\sum_{k=1}^{\infty} \frac{(2-2\alpha)_{k-1}}{k!} z^k \right) \\
&:= \left(z\sigma_n^{\beta} * \tau_{\alpha} \right)(z).
\end{aligned}
$$

Since $z\tau_{\alpha}'(z) = z(1-z)^{-(2-2\alpha)} \in \mathcal{S}^*(\alpha)$, Alexander's theorem implies that $\tau_{\alpha} \in \mathcal{C}(\alpha)$, and hence Lemma 3.1 yields $z\mathcal{G}_{\alpha,\beta} \in \mathcal{K}(\alpha)$. From Theorem 2.1, we deduce that $\mathcal{G}_{\alpha,\beta} * f'$ is zero-free in $\mathbb{D}$. □

Remark 3.1 For $\alpha = 1/2$, simple computations show that $(\mathcal{G}_{1/2,\beta} * f')(z) = (\sigma_n^{\beta} * f/z)(z) = \sigma_n^{\beta}(z, f/z)$. If $f \in \mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$, it follows from Corollary 3.1 that $\sigma_n^{\beta}(z, f/z) \ne 0$ in $\mathbb{D}$. This is a result of Ruscheweyh [11].

The next example relates to the Lerch transcendental function $\Phi(z,s,a)$ [14–16] given by

$$
\Phi(z,s,a) = \sum_{k=0}^{\infty} \frac{z^k}{(k+a)^s},
$$

$z \in \mathbb{D}$, $\operatorname{Re} s > 0$ and $a \in \mathbb{C}\setminus\{0,-1,-2,\ldots\}$. For $\operatorname{Re} s < 0$, the summand $(k+a)^{-s}$ can be continuously extended to $a = -k$, and in this case, $\Phi(z,s,a)$ is defined for all $a \in \mathbb{C}$.

Lemma 3.2 [13, Theorem 5.5] *Let $f \in \mathcal{C}(\alpha)$, $1/2 \le \alpha < 1$, and*

$$
\begin{aligned}
Q(z) &= \frac{1}{(n+1)^{\gamma}} \sum_{k=0}^{n} \frac{z^k}{(n+1-k)^{-\gamma}} \\
&= \frac{(-1)^{\gamma}}{(n+1)^{\gamma}} \left(\Phi(z,-\gamma,-n-1) - z^{n+1}\Phi(z,-\gamma,0) \right),
\end{aligned}
\tag{5}
$$

$n \in \mathbb{N}$, $\gamma \ge 0$. Then $zQ * f \in \mathcal{K}(\alpha)$.

The following result is evident from Lemma 3.2 and Theorem 2.1, and the details are therefore omitted.

Corollary 3.2 *Let $f \in \mathcal{R}_{\alpha}$, $1/2 \le \alpha < 1$. For $\gamma \ge 0$, let*

$$
\mathcal{H}_{\alpha,\gamma}(z) := \frac{1}{(n+1)^{\gamma}} \sum_{k=0}^{n} \frac{(n+1-k)^{\gamma}(2-2\alpha)_k}{(k+1)!} z^k = (Q * h)(z),
$$

where h is given by (4) and Q by (5). Then $\mathcal{H}_{\alpha,\gamma} * f'$ is zero-free in $\mathbb{D}$.

Remark 3.2 Now let $a_k := (n+1-k)^{\gamma}/(n+1)^{\gamma}$, $\gamma \geq 0$, $k = 0, 1, \ldots, n$. A computation shows that $1 = a_0 \geq a_1 \geq \cdots \geq a_n = 1/(n+1)^{\gamma} > 0$. For $\alpha = 1/2$, Corollary 3.2 yields

$$\left(\mathcal{H}_{1/2,\gamma} * f' \right)(z) = \left(\sum_{k=0}^{n} a_k z^k \right) * \left(\frac{f(z)}{z} \right) \neq 0.$$

This is Ruscheweyh result [11, Theorem 1, p.682], obtained in his work on the extension of the classical Kakeya-Eneström theorem. For $\alpha > 1/2$, Corollary 3.2 asserts more. If now

$$b_k := \frac{(n+1-k)^{\gamma}}{(n+1)^{\gamma}} \frac{(2-2\alpha)_k}{k!}, \quad \gamma \geq 0, \alpha \in (1/2, 1),$$

then $1 = b_0 \geq b_1 \geq \cdots \geq b_n = (2-2\alpha)_n/(n!(n+1)^{\gamma}) > 0$ and

$$\left(\mathcal{H}_{\alpha,\gamma} * f' \right)(z) = \left(\sum_{k=0}^{n} b_k z^k \right) * \left(\frac{f(z)}{z} \right) \neq 0.$$

Thus, the approximant is zero-free in $\mathbb{D}$ in spite of the fact that f may not be univalent (see Lemma 1.1(iv)).

For $\alpha \leq 1/2$, Lewis [1, Lemma 3, p.1118] proved that

$$q_n^{\alpha}(z) = \frac{n!}{(2-2\alpha)_n} \sum_{k=0}^{n} \frac{(2-2\alpha)_k}{k!} \frac{(2-2\alpha)_{n-k}}{(n-k)!} z^k$$

is the derivative of a function in $\mathcal{K}(\alpha)$. The polynomial q_n^{α} is related [1, p.1118] to the Jacobi polynomial $P_n^{a,b}(x) := \frac{(1+a)_n}{n!} {}_2F_1(-n, n+a+b+1; 1+a; (1-x)/2)$, $x \in [-1, 1]$, by

$$q_n^{\alpha}\left(e^{i\theta} \right) = \frac{n!(4-4\alpha)_n}{(2-2\alpha)_n(5/2-2\alpha)_n} e^{in\theta/2} P_n^{(3/2-2\alpha, 3/2-2\alpha)}\left(\cos(\theta/2) \right), \quad 0 \leq \theta \leq 2\pi.$$

Here ${}_2F_1$ is the Gaussian hypergeometric function [17].

Consider now the polynomial

$$\mathcal{Q}_{n,\alpha}(z) := \frac{n!}{(2-2\alpha)_n} \sum_{k=0}^{n} \frac{(2-2\alpha)_k}{k!} \frac{(2-2\alpha)_{n-k}}{(n-k)!} \frac{z^k}{k+1}. \tag{6}$$

A computation gives $(z\mathcal{Q}_{n,\alpha})' = q_n^{\alpha}$, and, thus, $z\mathcal{Q}_{n,\alpha} \in \mathcal{K}(\alpha)$, $\alpha \leq 1/2$. The following result is now easily derived from Theorem 2.1.

Corollary 3.3 [6, Theorem 4, p.122] *Let $f \in \mathcal{R}_{\alpha}$, $\alpha \leq 1/2$, and $\mathcal{Q}_{n,\alpha}$ be given by* (6). *Then $\mathcal{Q}_{n,\alpha} * f'$ is zero-free in $\mathbb{D}$.*

We next turn to consider zero-free non-polynomial approximants. Robinson [18] (also see [19, p.301]) introduced the polynomial

$$\mathcal{I}_n^{\beta}(z) := 1 + \sum_{k=1}^{n} \prod_{j=0}^{k-1} \frac{n-j}{\beta+n+j} z^k = {}_2F_1(1, -n; n+\beta; -z),$$

and conjectured that $z\mathcal{I}_n^\beta \in \mathcal{R}_{(3-\beta)/2}$, whenever $\beta \geq 1$ and $n \in \mathbb{N}$. Ruscheweyh and Salinas [20] resolved the conjecture with the following more general result.

Lemma 3.3 [20, Theorem 3, p.550] *Let* $\lambda \geq -1$ *and* $\beta \geq \max\{1, -2\lambda\}$. *Then* $z\mathcal{I}_\lambda^\beta(z) = z_2F_1(1, -\lambda; \lambda + \beta; -z) \in \mathcal{R}_{(3-\beta)/2}$.

A consequence of Lemma 3.3 is that $z\mathcal{I}_{\lambda-1}^3 = ((\lambda + 1)/\lambda)V_\lambda \in \mathcal{R}_0 = \mathcal{C}$ for $\lambda \geq -1/2$, where $V_\lambda(z) := \lambda/(\lambda + 1)z_2F_1(1, 1 - \lambda; \lambda + 2; -z)$, $\lambda > 0$ is a continuous extension (see [21]) of the de la Vallée Poussin means. Lemma 3.3 and Lemma 1.1(ii) together imply that $z\mathcal{I}_{\lambda-1}^{\beta+2} \in \mathcal{C} \subset \mathcal{K}$ for $\lambda > 0$ and $\beta \geq 1$. Theorem 2.1 now gives a non-polynomial approximant for outer functions.

Corollary 3.4 *If* $f \in \mathcal{C}$, *then* $\mathcal{I}_{\lambda-1}^{\beta+2}(z, f') = (\mathcal{I}_{\lambda-1}^{\beta+2} * f')(z)$ *is zero-free in* $\mathbb{D}$ *for all* $\lambda > 0$ *and* $\beta \geq 1$.

Remark 3.3 From [20], it is known that $\lim_{\lambda \to \infty} \mathcal{I}_\lambda^\beta(z) * f(z) = f(z)$. So if $f \in \mathcal{C}$ is bounded, then Corollary 3.4 implies that $\mathcal{I}_{\lambda-1}^{\beta+2}(z)$ is an approximant to the outer function f'. Thus, outer functions could also have zero-free non-polynomial approximants.

The following result on the prestarlikeness of functions, connected to the Gaussian hypergeometric function, will be required to prove the next theorem.

Lemma 3.4 [9, Theorem 2.12] *Let* $a, b \in \mathbb{R}$ *satisfy* $2b + 1 \geq |2a + 1|$. *Then*

$$z_2F_1(1, 1 + a, 1 + b, z) \in \mathcal{R}_{\frac{1-a-b}{2}}.$$

Theorem 3.1 *Let* $b \geq 1/2$ *and* $-b \leq a \leq 1 - b$. *Then the Cesàro means of order* $(a + b)$ *for the function* $_2F_1(1 + a + b, 1 + a; 1 + b; z)$ *is zero-free in* $\mathbb{D}$.

Proof Let $\alpha = (1 - a - b)/2$. Under the given hypothesis, it is evident that $0 \leq \alpha \leq 1/2$. The Cesàro means of order $a + b$ for the function $_2F_1(1 + a + b, 1 + a; 1 + b; z)$ can be expressed in the form

$$\sigma_n^{a+b}(z) * {}_2F_1(1 + a + b, 1 + a; 1 + b; z)$$

$$= \left(\sum_{k=0}^{n} \frac{n!}{(2 - 2\alpha)_n} \frac{(2 - 2\alpha)_{n-k}}{(n - k)!} \frac{(2 - 2\alpha)_k}{k!} \frac{z^k}{k + 1} \right) * \left(\sum_{k=0}^{\infty} \frac{(1 + a)_k}{(1 + b)_k} (k + 1)z^k \right)$$

$$= (g * f')(z),$$

where $g(z) = \mathcal{Q}_{n,\alpha}(z)$ is given by (6) and $f(z) = z_2F_1(1, 1 + a; 1 + b, z)$. It is known [1] that $zg = z\mathcal{Q}_{n,\alpha} \in \mathcal{K}(\alpha)$ for $\alpha \leq 1/2$. Straightforward computations show that $2b + 1 \geq |2a + 1|$, and, thus, Lemma 3.4 yields $f \in \mathcal{R}_\alpha$. Therefore, it follows from Theorem 2.1 that $g * f' \neq 0$ in $\mathbb{D}$. □

Example 3.1

(1) Choosing $a = b = 1/2$, Theorem 3.1 yields $\sigma_n^1(z, (1 - z)^{-2})$ is zero-free in $\mathbb{D}$.
(2) Since $_2F_1(1 + b, 1; 1 + b; z) = (1 - z)^{-1}$, with $a = 0$, it follows that $\sigma_n^b(z) \neq 0$ for $b \in [1/2, 1]$ and $z \in \mathbb{D}$.

(3) If $b = -a = 1/2$, Theorem 3.1 shows that the nth partial sum of the Taylor series of arctanh$(\sqrt{z})/\sqrt{z}$ is zero-free in $\mathbb{D}$.

When both the source functions f and the approximant are prestarlike of certain order, the result below shows that the range of the approximant satisfies a sector-like condition on the boundary.

Theorem 3.2 *Let $f \in \mathcal{R}_\alpha$ and $zg \in \mathcal{R}_\mu$ with $(zg * f)/z$ bounded in $\mathbb{D}$, $\alpha, \mu \leq 1/2$. Then the range of $g * f'$ is contained in a sector (from 0) with the opening $2\gamma\pi$ for some $\gamma < 1$.*

Proof Let $f \in \mathcal{R}_\alpha$, $\alpha \leq 1/2$. By Lemma 1.1(ii), $f \in \mathcal{R}_\alpha \subset \mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$. Rewrite $g * f'$ as

$$(g * f')(z) = \frac{1}{z}(z(zg)' * f)(z)$$
$$= \frac{1}{z}\left(zg(z)\frac{z(zg)'(z)}{zg(z)} * f(z)\right).$$

Since $zg \in \mathcal{R}_\mu \subset \mathcal{R}_{1/2} = \mathcal{S}^*(1/2)$, there exists a function $p \in \mathcal{P}(1/2)$ satisfying

$$\frac{z(zg)'(z)}{zg(z)} = p(z).$$

From Lemma 1.2, there exists a function $p_1 \in \mathcal{P}(1/2)$ such that

$$(g * f')(z) = \frac{(zg * f)(z)}{z}p_1(z).$$

Since $zg \in \mathcal{S}^*(1/2)$, and $f \in \mathcal{S}^*(1/2)$, Lemma 1.1(i) implies that $zg * f \in \mathcal{S}^*(1/2)$.

A result in [22, Theorem 2.6a, p.57] shows that

$$\frac{1}{z}(zg * f)(z) \in \mathcal{P}(1/2).$$

Since $(zg * f)/z$ is bounded in $\mathbb{D}$, there exists a γ_1 such that

$$\left|\arg\frac{(zg * f)(z)}{z}\right| < \frac{\gamma_1\pi}{2}, \quad \gamma_1 < 1.$$

Therefore,

$$\left|\arg(g * f')(z)\right| = \left|\arg\left\{\frac{(zg * f)(z)}{z}p_1(z)\right\}\right|$$
$$\leq \frac{\gamma_1\pi}{2} + \frac{\pi}{2} = \gamma\pi, \quad \gamma := (\gamma_1 + 1)/2 < 1.$$

$\square$

Example 3.2 Let g be either $\mathcal{I}_n^{\beta+1}$ or $\sigma_n^{\beta+1}$. The polynomial $(zg * f)/z$ is bounded in $\mathbb{D}$. Hence for $\beta \geq 1$, Theorem 3.2 implies that the range of both $\mathcal{I}_n^{\beta+1}(z, f')$ and $\sigma_n^{\beta+1}(z, f')$ are contained in a sector with opening $2\gamma\pi$, $\gamma < 1$.

Remark 3.4 Example 3.2 reduces to a result of Swaminathan [6, Theorem 3, p.121] in the case $f \in \mathcal{R}_0 = \mathcal{C}$ and $g(z) = \sigma_n^{\beta+1}(z)$.

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